Seat No. : _______________

DB-147

December-2018

M.Sc., Sem.-I

402 : Mathematics
(Measure & Integration)
(New)

Time : 2:30 Hours] **[Max. Marks : 70**

1. (A) (i) True or False ? If G is an open subset of [a, b] and $|G| = 0$, then $G = \phi$. (Give details). **7**

(ii) Show that $E \subset [a, b]$ is measurable if and only if given $\varepsilon < 0$ there exist a closed set $F \subset E$ and an open set $G \supset E$ such that $|G| - |F| < \varepsilon$. **7**

OR

(i) True or False ? If F is a closed subset of [a, b] and $|F| = 0$, then $F = \phi$. (Give details).

(ii) If $E \subset [a, b]$ show that there exists a subset H of E such that H is of type $F\sigma$ and $\underline{m}\, H = \underline{m}\, E$. (i.e. H and E have the same inner measure).

(B) Do any **four** : **4**

(i) Let S be the subset of [0, 6] given by S = (2, 3) U (3, 5]. Find the outer measure of S. (Do not prove)

(ii) Let S be the subset of [0, 10] given by S = (3, 8]. Find the inner measure of S. (Do not prove)

(iii) Find the length of the open set $\bigcup_{n=1}^{\infty} \left(\frac{1}{2^{n+1}}, \frac{1}{2^n} \right)$.

(iv) True or False ? Every open subset of [a, b] is measurable. (Do not prove)

(v) True or False ? Every closed subset of [a, b] is measurable. (Do not prove)

(vi) Give an example of a set $E \subset [0, 1]$, such that $mE = 0$. (Do not prove)

2. (A) (i) Suppose E_1 and E_2 are subsets of [a, b]. Further suppose that the symmetric difference of E_1 and E_2 has measure zero. Show that if E_1 is measurable, then E_2 is measurable. **7**

Also show that $mE_2 = mE_1$.

(ii) Let $f(x) = \frac{1}{x} \quad (0 < x < 1)$, **7**

$f(0) = 5, \quad f(1) = 7$

Prove that f is measurable on [0, 1]

OR

(i) If E_1 and E_2 are measurable subsets of [0, 1] and if $mE_1 = 1$, prove that $m(E_1 \cap E_2) = mE_2$.

(ii) Suppose that the function f on [a, b] is measurable. Show that for every $S \in \mathbb{R}$, the set $\{x \mid f(x) \geq s\}$ is a measurable set.

(B) Do any **four** : **4**

(i) True or False ? The union of uncountably many measurable subsets of [a, b] must be measurable. (Do not prove)

(ii) Give an example of an uncountable subset E of [0, 1] such that E is measurable. (Do not prove)

(iii) Consider the subsets $E_1 = \left(\frac{1}{2}, \frac{3}{4}\right)$ and $E_2 = \left(\frac{1}{4}, \frac{2}{3}\right)$ of [0, 1]. Find the measure of the symmetric difference of E_1 and E_2.

(iv) Let $S = \left\{x \in [0,1] \mid x^2 > \frac{1}{4}\right\}$. Find the measure of S.

(v) Suppose f is a measureable function on [0, 1], and let g be defined by $g(x) = f(x), x \notin \left\{0, \frac{1}{2}, 1\right\}$. $g(0) = g\left(\frac{1}{2}\right) = g(1) = 5$. Is g measurable ?

(vi) Let f, g be functions defined on [0, 2] as follows :

$f(x) = x, g(x) = x^2$.

Draw the graph of the function max (f, g).

3. (A) (i) Show that if f is a bounded measureable function on [a, b], then f is Lebesgue integrable. **7**

(ii) Let E_1 and E_2 be measurable subsets of [0, 1]. Suppose $E_1 \cup E_2 = [0, 1]$, show that at least one of the sets E_1, E_2 has measure $\geq \frac{1}{2}$. **7**

OR

(i) Let f be defined on [0, 1] by $f(x) = x$. Let E_1 be the inverse image under f of $\left[0, \frac{1}{2}\right]$. Let E_2 be the inverse image under f of $\left[\frac{1}{2}, \frac{3}{4}\right]$. Let E_3 be the inverse image under f of $\left[\frac{3}{4}, 1\right]$. Show that $P = \{E_1, E_2, E_3\}$ is a measureable partition of [0, 1]. Calculate U [f; p] and L [f; p].

(ii) Suppose E_1 and E_2 are measureable subsets of [a, b] and f is a bounded measurable (so Lebesgue integrable) function defined on [a, b]. Prove that $\int_{E_1} f + \int_{E_2} f = \int_{E_1 \cup E_2} f + \int_{E_1 \cap E_2} f$.

(B) Do any **three** : **3**

(i) Suppose $E \subset [0, 1]$ is measurable and $mE = \frac{1}{2}$. Find $\int_E 1$. (Do not prove)

(ii) Suppose f is defined on [0, 1] by $f(x) = x$. Suppose $E = \left(\frac{1}{2}, \frac{3}{4}\right)$. Find $\int_E f$.

(iii) Suppose E_1 and E_2 are disjoint measurable subsets of [a, b] and f is a bounded function in $\mathcal{L}$ [a, b]. Suppose $\int_{E_1} f = 2$ and $\int_{E_2} f = 3$. Find $\int_{E_1 \cup E_2} f$.

(Do not prove)

(iv) Let $f(x) = 0$ if x is rational, $x \in [0, 1]$ and $f(x) = 1$ if x is irrational, $x \in [0, 1]$. Find $\int_0^1 f$. (Do not prove).

(v) Evaluate $\int_0^{\pi} f$, where $f(x) = \sin x$, $x \in [0, \pi]$.

4. (A) (i) Let $f(x) = \frac{1}{x^p}$, $(0 < x \leq 1)$ **7**

Prove that $f \in \mathcal{L}[0, 1]$, if $p < 1$.

Find the value of $\int_0^1 f$, if $p < 1$.

(ii) If $f \in \mathcal{L}[a, b]$ and if $F(x) = \int_a^x f(t)\,dt$, $(a \leq x \leq b)$. Prove that F is continuous on [a, b]. **7**

OR

(i) State Fatou's lemma. (Do not prove)

(ii) Let $f \in \mathcal{L}[a, b]$. Let $\varepsilon > 0$ be given. Show that there exists a $\delta > 0$ such that $\left|\int_E f\right| < \varepsilon$ whenever E is a measurable subset of [a, b] with $mE < \delta$.

(B) Do any **three** : **3**

(i) If $f(x) = \frac{1}{x}$ $(0 < x \leq 1)$, find ^2f.

(ii) If $f(x) = \sin x$ $(0 \leq x \leq 2\pi)$, draw the graph of f^+.

(iii) True or False ? If $f(x) = g(x)$ almost everywhere $(x \in E)$ then $^5f(x) = {}^5g(x)$ almost everywhere $(x \in E)$. (Do not prove)

(iv) True or False ? If f and g are in $\mathcal{L}[a, b]$, then $fg \in \mathcal{L}[a, b]$. (Do not prove).

(v) State (without proof) the Lebesgue Dominated Convergence Theorem.